

# Lecture 09: Shamir Secret Sharing (Lagrange Interpolation)

## Recall: Goal

We want to

Share a secret  $s \in \mathbb{Z}_p$  to  $n$  parties, such that  $\{1, \dots, n\} \subseteq \mathbb{Z}_p$ ,

Any two parties can reconstruct the secret  $s$ , and

No party alone can predict the secret  $s$

## Recall: Secret Sharing Algorithm

SecretShare( $s, n$ )

Pick a random line  $\ell(X)$  that passes through the point  $(0, s)$

This is done by picking  $a_1$  uniformly at random from the set  $\mathbb{Z}_p$

And defining the polynomial  $\ell(X) = a_1X + s$

Evaluate  $s_1 = \ell(X = 1)$ ,  $s_2 = \ell(X = 2)$ ,  $\dots$ ,  $s_n = \ell(X = n)$

Secret shares for party 1, party 2,  $\dots$ , party  $n$  are  $s_1, s_2, \dots, s_n$ , respectively

# Recall: Reconstruction Algorithm

SecretReconstruct( $i_1, s^{(1)}, i_2, s^{(2)}$ )

Reconstruct the line  $\ell'(X)$  that passes through the points  $(i_1, s^{(1)})$  and  $(i_2, s^{(2)})$

We will learn a new technique to perform this step, referred to as the Lagrange Interpolation

Define the reconstructed secret  $s' = \ell'(0)$

# General Goal

We want to

Share a secret  $s \in \mathbb{Z}_p$  to  $n$  parties, such that  $\{1, \dots, n\} \subseteq \mathbb{Z}_p$ ,

Any  $t$  parties can reconstruct the secret  $s$ , and

Less than  $t$  parties cannot predict the secret  $s$

# Shamir's Secret Sharing Algorithm

SecretShare( $s, n$ )

Pick a polynomial  $p(X)$  of degree  $\leq (t - 1)$  that passes through the point  $(0, s)$

This is done by picking  $a_1, \dots, a_{t-1}$  independently and uniformly at random from the set  $\mathbb{Z}_p$

And defining the polynomial

$$\ell(X) = a_{t-1}X^{t-1} + a_{t-2}X^{t-2} + \dots + a_1X + s$$

Evaluate  $s_1 = p(X = 1), s_2 = p(X = 2), \dots, s_n = p(X = n)$

Secret shares for party 1, party 2,  $\dots$ , party  $n$  are  $s_1, s_2, \dots, s_n$ , respectively

# Shamir's Reconstruction Algorithm

SecretReconstruct( $i_1, s^{(1)}, i_2, s^{(2)}, \dots, i_t, s^{(t)}$ )

Use Lagrange Interpolation to construct a polynomial  $p'(X)$  that passes through  $(i_1, s^{(1)}), \dots, (i_t, s^{(t)})$  (we describe this algorithm in the following slides)

Define the reconstructed secret  $s' = p'(0)$

# Lagrange Interpolation: Introduction I

Consider the example we were considering in the previous lecture

The secret was  $s = 3$

Secret shares of party 1, 2, 3, and 4, were 0, 2, 4, and 1, respectively

Suppose party 2 and party 3 are trying to reconstruct the secret

Party 2 has secret share 2, and

Party 3 has secret share 4

We are interested in finding the line that passes through the points  $(2, 2)$  and  $(3, 4)$



# Lagrange Interpolation: Introduction II

## Subproblem 1:

Let us find the line that passes through  $(2, 2)$  and  $(3, 0)$

Note that at  $X = 3$  this line evaluates to 0, so

$X = 3$  is a root of the line

So, the line has the equation  $l_1(X) = c \cdot (X - 3)$ ,  
where  $c$  is a suitable constant

Now, we find the value of  $c$  such that  $l_1(X)$  passes  
through the point  $(2, 2)$

So, we should have  $c \cdot (2 - 3) = 2$ , i.e.,  $c = 3$

$l_1(X) = 3 \cdot (X - 3)$  is the equation of that line

## Subproblem 2:

Let us find the line that passes through  $(2, 0)$  and  $(3, 4)$

Note that at  $X = 2$  this line evaluates to 0, so  
 $X = 2$  is a root of the line

So, the line has the equality  $l_2(X) = c \cdot (X - 2)$ ,  
where  $c$  is a suitable constant

Now, we find the value of  $c$  such that  $l_2(X)$  passes  
through the point  $(3, 4)$

So, we should have  $c \cdot (3 - 2) = 4$ , i.e.  $c = 4$

$$l_2(X) = 4 \cdot (X - 2)$$

Putting Things Together:

Define  $\ell'(X) = \ell_1(X) + \ell_2(X)$

That is, we have

$$\ell'(X) = 3 \cdot (X - 3) + 4 \cdot (X - 2)$$

Evaluation of  $\ell'(X)$  at  $X = 0$  is

$$s' = \ell'(X = 0) = 3 \cdot (-3) + 4 \cdot (-2) = 3 \cdot 2 + 4 \cdot 3 = 1 + 2 = 3$$

# Uniqueness of Polynomial I

We shall prove the following result

## Theorem

*There is a unique polynomial of degree at most  $d$  that passes through  $(x_1, y_1), (x_2, y_2), \dots, (x_{d+1}, y_{d+1})$*

If possible, let there exist two distinct polynomials of degree  $\leq d$  such that they pass through the points  $(x_1, y_1), (x_2, y_2), \dots, (x_{d+1}, y_{d+1})$

Let the first polynomial be

$$p(X) = a_d X^d + a_{d-1} X^{d-1} + \dots + a_1 X + a_0$$

Let the second polynomial be

$$p'(X) = a'_d X^d + a'_{d-1} X^{d-1} + \dots + a'_1 X + a'_0$$

# Uniqueness of Polynomial II

Let  $p^*(X)$  be the polynomial that is the difference of the polynomials  $p(X)$  and  $p'(X)$ , i.e.,

$$p^*(X) = p(X) - p'(X) = (a_d - a'_d)X^d + \dots + (a_1 - a'_1)X + (a_0 - a'_0)$$

**Observation.** The degree of  $p^*(X)$  is  $\leq d$

# Uniqueness of Polynomial III

For  $i \in \{1, \dots, d + 1\}$ , note that at  $X = x_i$  both  $p(X)$  and  $p'(X)$  evaluate to  $y_i$

So, the polynomial  $p^*(X)$  at  $X = x_i$  evaluates to  $y_i - y_i = 0$ , i.e.  $x_i$  is a root of the polynomial  $p^*(X)$

**Observation.** The polynomial  $p^*(X)$  has roots  $X = x_1$ ,  $X = x_2, \dots, X = x_{d+1}$

# Uniqueness of Polynomial IV

We will use the following result

## Theorem (Schwartz–Zippel, Intuitive)

*A non-zero polynomial of degree  $d$  has at most  $d$  roots (over any field)*

### Conclusion.

Based on the two observations above, we have a  $\leq d$  degree polynomial  $p^*(X)$  that has at least  $(d + 1)$  distinct roots  $x_1, \dots, x_{d+1}$

This implies, by Schwartz–Zippel Lemma, that the polynomial is the zero-polynomial.

That is,  $p^*(X) = 0$ .

This implies that  $p(X)$  and  $p'(X)$  are identical

This contradicts the initial assumption that there are two distinct polynomials  $p(X)$  and  $p'(X)$

# Summary

The proof in the previous slides proves that

Given a set of points  $(x_1, y_1) , \dots , (x_{d+1}, y_{d+1})$

There is a unique polynomial of degree at most  $d$  that passes through all of them!



# Lagrange Interpolation I

Suppose we are interested in constructing a polynomial of degree  $\leq d$  that passes through the points  $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$

## Subproblem $i$ :

We want to construct a polynomial  $p_i(X)$  of degree  $\leq d$  that passes through  $(x_i, y_i)$  and  $(x_j, 0)$ , where  $j \neq i$

So,  $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{d+1}\}$  are roots of the polynomial  $p_i(X)$

Therefore, the polynomial  $p_i(X)$  looks as follows

$$p_i(X) = c \cdot (X - x_1) \cdots (X - x_{i-1})(X - x_{i+1}) \cdots (X - x_{d+1})$$

Tersely, we will write this as

$$p_i(X) = c \cdot \prod_{\substack{j \in \{1, \dots, d+1\} \\ \text{such that } j \neq i}} (X - x_j)$$

## Lagrange Interpolation III

Now, to evaluate  $c$  we will use the property that

$$p_i(x_i) = y_i$$

Observe that the following value of  $c$  suffices

$$c = \frac{y_i}{\prod_{\substack{j \in \{1, \dots, d+1\} \\ \text{such that } j \neq i}} (x_i - x_j)}$$

So, the polynomial  $p_i(X)$  that passes through  $(x_i, y_i)$  and  $(x_j, 0)$ , where  $j \neq i$  is

$$p_i(X) = \frac{y_i}{\prod_{\substack{j \in \{1, \dots, d+1\} \\ \text{such that } j \neq i}} (x_i - x_j)} \cdot \prod_{\substack{j \in \{1, \dots, d+1\} \\ \text{such that } j \neq i}} (X - x_j)$$

Observe that  $p_i(X)$  has degree  $d$

## Putting Things Together:

Consider the polynomial

$$p(X) = p_1(X) + p_2(X) + \dots + p_{d+1}(X)$$

This is the desired polynomial that passes through  $(x_i, y_i)$

### Claim

*The polynomial  $p(X)$  passes through  $(x_i, y_i)$ , for  $i \in \{1, \dots, d + 1\}$*

# Lagrange Interpolation V

Proof.

Note that, for  $j \in \{1, \dots, d + 1\}$ , we have

$$p_j(x_i) = \begin{cases} y_i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

Therefore,  $p(x_i) = \sum_{j=1}^{d+1} p_j(x_i) = y_i$



# Summary of Interpolation

Given points  $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$

Lagrange Interpolation provides one polynomial of degree  $\leq d$  polynomial that passes through all of them

Theorem 1 states that this  $\leq d$  degree polynomial is unique

# Example for Lagrange Interpolation I

Let us find a degree  $\leq 2$  polynomial that passes through the points  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$

Subproblem 1:

We want to find a degree  $\leq 2$  polynomial that passes through the points  $(x_1, y_1)$ ,  $(x_2, 0)$ , and  $(x_3, 0)$

The polynomial is

$$p_1(X) = \frac{y_1}{(x_1 - x_2)(x_1 - x_3)}(X - x_2)(X - x_3)$$

## Example for Lagrange Interpolation II

Subproblem 2:

We want to find a degree  $\leq 2$  polynomial that passes through the points  $(x_1, 0)$ ,  $(x_2, y_2)$ , and  $(x_3, 0)$ .

The polynomial is

$$p_2(X) = \frac{y_2}{(x_2 - x_1)(x_2 - x_3)}(X - x_1)(X - x_3)$$

Subproblem 3:

We want to find a degree  $\leq 2$  polynomial that passes through the points  $(x_1, 0)$ ,  $(x_2, 0)$ , and  $(x_3, y_3)$ .

The polynomial is

$$p_2(X) = \frac{y_3}{(x_3 - x_1)(x_3 - x_2)}(X - x_1)(X - x_2)$$



## Example for Lagrange Interpolation III

Putting Things Together: The reconstructed polynomial is

$$p(X) = p_1(X) + p_2(X) + p_3(X)$$

# Conclusion

This completes the description of Shamir's Secret Sharing algorithm. In the following lectures we will argue its security.